NBU-003-1261001 Seat No. ________

M. Phil. (Science) (Sem. I) (CBCS) Examination

April / May - 2017

Mathematics : CMT-10001

(Algebra) (New Course)

Faculty Code : 003
Subject Code : 1261001

Time : $2\frac{1}{2}$ Hours] [Total Marks : **70**

Instructions : (1) Answer all the questions.
(2) Each question carries 14 marks.
(3) The rings considered here are commutative with identify.

1 Answer : (any seven) **7×2=14**

(a) Define a prime ideal of a ring. Verify that $35\mathbb{Z}$ is not a prime ideal of $\mathbb{Z}$.
(b) When is an ideal of a ring said to be finitely generated ?
(c) Define an Artinian ring.
(d) Define a multiplicatively closed subset of a ring and illustrate it with an example
(e) Define :
  (i) nilradical of a ring and
  (ii) Jacobson radical of a ring.
(f) Define a primary ideal of a ring.
(g) Prove that $2+i\sqrt{7}$ is integral over $\mathbb{Z}$.
(h) State going-up theorem.
(i) Define a valuation ring of a field *K*. Verify that $\mathbb{Z}_{5\mathbb{Z}}$ is a valuation ring of $\mathbb{Q}$.
(j) State the second uniqueness theorem for decomposable submodules of a module *M* over a ring *R*.

2 Answer any two : **2×7=14**

(a) Let I be an ideal of a ring R such that $I \neq R$. Show that there exists a maximal ideal m of R such that $I \subseteq m$.

(b) State and prove Nakayana's lemma.

(c) Let I be an ideal of a ring R. Let C be the collection of all prime ideals p of R such that $p \supseteq I$. Show that $\sqrt{I} = \cap_{p \in cp}$.

3 (a) Let $f : R \to T$ be a homomorphism of rings. Let I_1, I_2 be ideals of R. Prove : **5**

(i) $(I_1 + I_2)^e = I_1^e + I_2^e$ and

(ii) $(I_1 \cap I_2)^e \subseteq I_1^e \cap I_2^e$.

(b) Let M be a module over a ring R. If $M_m = (0)$ for each maximal ideal m of R, then show that $M = (0)$. **5**

(C) Let R be an Artinian integral domain. Prove that R is a field. **4**

OR

3 (a) Let $N \subseteq K$ be submodules of a module M over a ring R. Prove that $(M/N)(K/N) \cong M/K$ as R-modules. **5**

(b) Show that the homomorphic image of a Noetherian ring is Noetherian. **5**

(c) Let I be an ideal of a ring R such that $\sqrt{I} = m$ is a maximal ideal of R. Prove that I is m-primary. **4**

4 Answer any two : **2×7=14**

(a) Prove that first uniqueness theorem on decomposable submodules of a module M over a ring R.

(b) Let S be a multiplicatively closed subset of a ring R. Let C = {p : p is a prime ideal of R such that $p \cap S = \theta$}. Let D be the collection of all prime ideals of $S^{-1}R$. Show that there is a bijection from C onto D.

(c) Let R be a subring of a ring T. Let $t \in T$. Suppose that there exists a faithful $R[t]$-module M such that M is a finitely generated R-module. Prove that t is integral over R.

5 Answer any **two** : **2×7=14**

(a) Let R be a Noetherian ring. Prove that the nilradical of R is nilpotent.

(b) Let S be a multiplicatively closed subset of a ring R. Let $g: R \rightarrow T$ be a ring homomorphism such that $g(s)$ is a unit in T for all $s \in S$. Show that there exists a unique ring homomorphism $h: S^{-1}R \rightarrow T$ such that $h(r/1) = g(r)$ for all $r \in R$.

(c) Let $I = q_1 \cap \ldots\ldots \cap \ldots\ldots \cap q_n$ be an irredundant primary decomposition of an ideal I of a ring R, where q_i is a p_i- primary ideal of R for each $i \in \{1, \ldots., n\}$ Prove that $Z_R(R/I) = \bigcup_{i=1}^{n} p_i$.

(d) Let R be a subring of a ring T such that T is integral over R. Let $p_1 \subseteq p_2$ be prime ideals of T such that $p_1 \cap R = p_2 \cap R$. Show that $p_1 = p_2$.
